

A MATHEMATICAL CLUE TO THE SEPARATION PHENOMENA ON THE TWO-DIMENSIONAL NAVIER-STOKES EQUATION

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ABSTRACT. In general, before separating from a boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction. Here in this paper, a clue to such reverse flow phenomena (in the mathematical sense) is observed. More precisely, the non-stationary two-dimensional Navier-Stokes equation with an initial datum having a parallel laminar flow (we define it rigorously in the paper) is considered. We show that the direction of the material differentiation is opposite to the initial flow direction and effect of the material differentiation (inducing the reverse flow) becomes bigger when the curvature of the boundary becomes bigger. We also show that the parallel laminar flow cannot be a stationary Navier-Stokes flow near a portion of the boundary with nonzero curvature.

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1. INTRODUCTION

Uchida, Sugitani and Ohya [14] studied a non-stratified airflow past a two-dimensional ridge in a uniform flow. Airflows around the ridge include an unsteady vortex shedding. Their study focused on airflow characteristics in a wake region. In general, before separating from a boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction (see [6, 14] for example). There are several results related to the wake region in pure mathematics. Using the Oseen system is one of the mathematical approach to analyze the wake region. For the detailed discussion of the Oseen system, we refer the reader to [4]. In a convex obstacle case, the character of the system is elliptic in front of the obstacle. To the contrary, its character changes into parabolic type (wake region) behind the obstacle (see [9] for example). Maekawa [12] considered the two-dimensional Navier-Stokes equations in a half plane under the no-slip boundary condition. He established a solution formula for the vorticity equations and got a sufficient condition on the initial data for the vorticity to blow up to the inviscid limit (see also [4]). His observation suggests that if the Reynolds number is high, the boundary layer immediately appears and the high vorticity creation occurs near the initial time and the boundary. Ma and Wang [11] provided a characterization of the boundary layer separation of 2-D incompressible viscous fluids. They considered a separation equation linking a separation location and a time with the Reynolds number, the external forcing and the initial velocity field. In the Dirichlet boundary condition case, which corresponds to the boundary layer separation, the above mentioned work of Ma and Wang provides detailed information on the flow transition near the critical time. However, none of the above studies has shown the mechanism behind the reverse flow phenomena rigorously. In this

paper we observe a mathematical clue to analyze such reverse flow phenomena. More precisely, we consider the non-stationary Navier-Stokes equations with an initial datum having a parallel laminar flow (we define rigorously later). We show that the direction of the material differentiation is opposite to the initial flow direction and effect of the material differentiation (inducing the reverse flow) becomes bigger when the curvature of the boundary becomes bigger. Before showing such result, we show the parallel laminar flow cannot be a stationary Navier-Stokes flow near a portion of the boundary with nonzero curvature.

Now we formulate our results in the following mathematical setting. Let Ω be a either bounded or unbounded domain with smooth boundary in \mathbb{R}^2 . The stationary Navier-Stokes equations are expressed as:

$$(1.1) \quad -\nu \Delta u + (u \cdot \nabla)u = -\nabla p \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega$$

with $u|_{\partial\Omega} = 0$. We need to handle a shape of the boundary $\partial\Omega$ precisely, thus we set a parametrized smooth boundary $\varphi : (-\infty, \infty) \mapsto \mathbb{R}^2$ as $|\partial_s \varphi(s)| = 1$, $\theta(\partial_s \varphi(s))$ is a decreasing function, $\cup_{-\infty < s < \infty} \varphi(s) \subset \partial\Omega$, where $\theta(w)$ is defined by a vector $w = r(\cos \tilde{\theta}, \sin \tilde{\theta})$, $\theta(w) := \tilde{\theta}$. For a technical sense, we need to assume that there are \bar{s}_0 and \bar{s}_2 (we set $\bar{s}_1 = 0$, $\bar{s}_0 < \bar{s}_1 < \bar{s}_2$) s.t. $\varphi(\bar{s}_0) = (0, 0)$, $\theta(\partial_s \varphi(s))|_{s=\bar{s}_0} = 0$, $|\partial_s^2 \varphi(s)|$ is monotone increasing for $s \in [\bar{s}_0, \bar{s}_1]$, $|\partial_s^2 \varphi(s)| = 1/\delta$ for $s \in [\bar{s}_1, \bar{s}_2]$, where $1/\delta$ is a constant curvature of a part of obstacle boundary $\cup_{s \in [\bar{s}_1, \bar{s}_2]} \varphi(s)$. Note that there are \tilde{s} and \tilde{x} , then $\varphi(s)$ ($s \in [\bar{s}_1, \bar{s}_2]$) can be expressed as

$$(1.2) \quad \varphi(s) = \delta \left(\cos \frac{(s + \tilde{s})}{\delta}, \sin \frac{(s + \tilde{s})}{\delta} \right) + \tilde{x}.$$

Here we mainly consider laminar type flows (we define the “laminar type flow” later) in a localized region near the boundary, thus we need to assume $u \neq 0$ (no stagnation point) in Ω near $\cup_{\bar{s}_0 < s < \bar{s}_2} \varphi(s)$. Boundary layers appear on the surface of bodies in viscous flow because the fluid seems to stick to the boundary $\partial\Omega$. Right at the surface the flow has zero relative speed and this fluid transfers momentum to adjacent layers through the action of viscosity. We need to express such phenomena in pure mathematics. To do so, we need to give the following coordinates.

Definition 1.1. (Normal coordinate.) For $s \in [\bar{s}_0, \bar{s}_2]$, let

$$\Phi(s, r) = \Phi_\varphi(s, r) := (\partial_s \varphi(s))^\perp r + \varphi(s).$$

We define \perp as the upward direction.

Remark 1.2. (An infinite channel with uniform width.) We give a typical example of Ω . Assume that there is $\bar{s}_3 (> \bar{s}_2)$ s.t. $|\partial_s^2 \varphi(s)|$ is monotone decreasing for $s \in [\bar{s}_2, \bar{s}_3]$ and $|\partial_s^2 \varphi(\bar{s}_0)| = |\partial_s^2 \varphi(\bar{s}_3)| = 0$. Assume moreover that Ω is a straight pipe except near the origin, namely, we assume that $\varphi(s) = (s, 0)$ for $s < 0$, $\theta(\partial_s \varphi(s))$ is a constant for $s > \bar{s}_3$ and $-\pi < \theta(\partial_s \varphi(\bar{s}_3)) < 0$. Then we find an infinite channel with uniform width as

$$\Omega = \{\Phi(s, r) : s \in (-\infty, \infty), r \in (0, 1)\}.$$

In this case, there are several results related to the Poiseuille flow (see [9] for example).

Remark 1.3. By a scaling argument with a cone shape (1.2), we see that for any $s_1, s_2 \in [\bar{s}_1, \bar{s}_2]$ with $s_1 < s_2$,

$$|\cup_{s'=s_1}^{s_2} \Phi(s', r_1)| = \left(\frac{r_1 + \delta}{r_2 + \delta} \right) |\cup_{s'=s_1}^{s_2} \Phi(s', r_2)|.$$

In order to state main theorems, we need to give several definitions.

Definition 1.4. (Normalized streamline and its direction.) Let γ_a be in Ω near $\cup_{\bar{s}_0 < s < \bar{s}_2} \varphi(s)$ which satisfies

$$\partial_s \gamma_a(s) = \left(\frac{u}{|u|} \right) (\gamma_a(s)), \quad \gamma_a(0) = a \in \Omega \quad \text{near} \quad \cup_{\bar{s}_0 < s < \bar{s}_2} \varphi(s).$$

Moreover, we assume that the flow is to the rightward direction (laminar flow direction), namely,

$$0 < \left\langle \partial_s \varphi(s), \frac{u}{|u|}(\Phi(s, r)) \right\rangle < 1$$

for $s \in [\bar{s}_0, \bar{s}_2]$ and sufficiently small r , where $\langle \cdot, \cdot \rangle$ means inner product.

We have already defined that there is no stagnation point near the boundary. Thus the above definition is well defined.

Definition 1.5. (Poincaré map.) For fixed s and s_1 sufficiently close to each other, let s_{min} be the minimum of $s' > 0$ for which there exists $\tau = \tau(s')$ such that $\Phi(s_1, \tau(s')) = \gamma_{\Phi(s, r)}(s')$. Let $L(r) = L_{s, s_1}(r) = \tau(s_{min})$.

Definition 1.6. (Classification of laminar type flows.) The velocity u near the boundary $\cup_{s' \in [\bar{s}_0, \bar{s}_2]} \varphi(s')$ is called:

“Strong diverging laminar flow” iff $L(r)/r > C > 1$ near the boundary,

“Weak diverging laminar flow” iff $L(r)/r \geq 1$ and $R(r)/r \rightarrow 1$ ($r \rightarrow 0$),

“Parallel laminar flow” iff $L(r)/r = 1$ near the boundary.

Remark 1.7. In the parallel laminar flow case, we see that $|u(\gamma_a(s))| = |u(a)|$ if a and $\gamma_a(s)$ are sufficiently close to $\cup_{s \in [\bar{s}_0, \bar{s}_2]} \varphi(s)$.

In this paper we only consider the parallel laminar flow case. In the parallel laminar flow case, we see that

$$(1.3) \quad \theta(\partial_s \gamma_{(0, x_2)}(s))|_{s=0} = 0 \quad \text{for sufficiently small } x_2 > 0.$$

In this case, it is natural to set

$$(1.4) \quad u_1(0, x_2) = \alpha_1 x_2 - \frac{\alpha_2}{2} x_2^2 (= h(x_2)) \quad \text{and} \quad u_2(0, x_2) = 0 \quad (\alpha_1, \alpha_2 > 0)$$

in order to satisfy (1.1) for sufficiently small $x_2 > 0$. In this case, we can say that the boundary layer thickness is α_1/α_2 . This setting is based on “Poiseuille flow” (see Remark 1.2 and [8] for example). Thus the parallel flow satisfying (1.4) is one of the candidate of the solution to the stationary Navier-Stokes equation.

Remark 1.8. We see from Remark 1.7 that

$$(1.5) \quad h(x_2) = |u(\Phi(s, x_2))|$$

if $\Phi(s, x_2)$ is sufficiently close to $\cup_{s \in [\bar{s}_0, \bar{s}_2]} \varphi(s)$.

The first main result is as follows:

Theorem 1.9. (*Stationary case.*) *Let us choose $\nu > 0$ arbitrarily and fix it. Then there is no parallel laminar flow satisfying the stationary Navier-Stokes equation (1.1) and (1.4) near $\cup_{s'=\bar{s}_1}^{\bar{s}_2} \varphi(s)$.*

Next we give a result in the non-stationary Navier-Stokes case. The non-stationary Navier-Stokes equations are expressed as

$$(1.6) \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u = -\nabla p, & \text{div } u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = 0 \quad \text{and} \quad u|_{t=0} = u_0. \end{cases}$$

Throughout the non-stationary case we assume that $p(x, t)|_{t=0}$ is smooth in space valuable. If u_0 is smooth and in $L^2(\Omega)$ (which means finite energy), then the solution (u, p) is a global-in-time smooth solution (see [10]). By Theorem 1.9, we see that if we take an initial datum which has a parallel laminar flow near the boundary, the solution u must be a non-stationary flow in some small time interval near the initial time. In this case, we observe a mathematical clue to the separation phenomena. More precisely, we see that the direction of the material differentiation is opposite to the initial flow direction and the effect of the material differentiation becomes bigger when the curvature of the boundary becomes bigger. Let be more precise. The initial datum having a parallel laminar flow satisfies

$$(1.7) \quad \left\langle \frac{u_0}{|u_0|}(x), \partial_s \varphi(s) \right\rangle \rightarrow 1 \quad \text{as } x \rightarrow \varphi(s) \quad (s \in [\bar{s}_1, \bar{s}_2]).$$

At some time $t > 0$, if the solution $u(x, t)$ has the reverse direction near the boundary against the laminar flow direction, we should have

$$(1.8) \quad \left\langle \frac{u}{|u|}(x, t), \partial_s \varphi(s) \right\rangle \rightarrow -1 \quad \text{as } x \rightarrow \varphi(s) \quad (s \in [\bar{s}_1, \bar{s}_2]) \quad \text{for some } t > 0.$$

To obtain such changing direction property, we can expect that the solution should satisfy

$$(1.9) \quad \frac{\langle D_t u(x, t), \partial_s \varphi(s) \rangle}{\langle u_0(x), \partial_s \varphi(s) \rangle} \leq C < 0 \quad \text{as } x \rightarrow \varphi(s) \quad (s \in [\bar{s}_1, \bar{s}_2])$$

for a long time interval up to the time t , where $D_t u := \partial_t u + (u \cdot \nabla)u$ is the material differentiation. We show that the solution $u(x, t)$ to the non-stationary Navier-Stokes equation satisfies (1.9) at the initial time, if the initial datum has a parallel laminar flow near the boundary. The following is the second main theorem.

Theorem 1.10. *Assume that $\nabla p(t, x)|_{t=0}$ is smooth. If the initial datum satisfies (1.4) and has “Parallel laminar flow”, then we have*

$$\lim_{x \rightarrow \varphi(s)} \frac{\langle D_t u(x, t)|_{t=0}, \partial_s \varphi(s) \rangle}{\langle u_0(x), \partial_s \varphi(s) \rangle} = -\frac{\nu \alpha_2}{\delta \alpha_1} - \frac{\nu}{\delta^2} < 0 \quad \text{for } s \in [\bar{s}_1, \bar{s}_2].$$

Recall that the value α_1/α_2 is defined as the boundary layer thickness. In general, the pressure is a non-local operator. However the values in the above theorem are depending only on the behavior of the flow near the boundary $\cup_{s \in [\bar{s}_1, \bar{s}_2]} \varphi(s)$.

Remark 1.11. There are direct and indirect evidences for the validity of the “Kutta condition” in restricted regions (see [2]). The method used in the above theorem may give another support for the validity of the Kutta condition in pure mathematical sense.

Remark 1.12. For the Poiseuille flow case, the pressure is depending on ν . More precisely, if

$$\Omega = \{x \in \mathbb{R}^2 : 0 < x_2 < 1\},$$

then $u = (x_2 - x_2^2, 0)$, $p = (\nu/2)x_2$ is one of the stationary solution to (1.1). To the contrary, for the whole space case (without any boundary effect), the pressure (in the sense of “mild solution”) is expressed as

$$p = \sum_{i,j=1,2} R_i R_j u_i u_j,$$

where R_j is the Riesz transform (see [5, 7, 13]). In this observation, the dependence on the viscosity appears due to the Dirichlet boundary.

2. PROOFS OF THE MAIN THEOREMS

Proof of the first main theorem. First we estimate Δu . For $\bar{s} \in [\bar{s}_1, \bar{s}_2]$ that is selected, we call the point $\varphi(\bar{s})$ to be Q . That is $Q = \varphi(\bar{s})$. Then, we use the orthonormal frame $e_1 = \partial_s \varphi|_{\bar{s}}$, $e_2 = (\partial_s \varphi)^\perp|_{\bar{s}}$ at the point Q to construct a cartesian coordinate system with the new x_1 -axis to be the straight line which passes through Q and parallel to the vector e_1 , and the new x_2 -axis to be the straight line which passes through Q and is parallel to the vector e_2 . Then, for the given vector field u in an open neighborhood near $\partial\Omega$, we define for each $y \in \Omega$ near $\partial\Omega$, the two components of $u(y)$ with respect to the e_1 direction and the e_2 direction. It is expressed as

$$u(y) = v^1(y)e_1 + v^2(y)e_2,$$

where $v^1(y) := \langle u(y), e_1 \rangle$ and $v^2(y) := \langle u(y), e_2 \rangle$. Since $\varphi([\bar{s}_1, \bar{s}_2])$ is known to be a circular arc with radius δ , let C to be the point at which the center of the circular arc $\varphi([\bar{s}_1, \bar{s}_2])$ is located. Let (s, r) to be the coordinate representation of the point y in the coordinate system *based at* Q which is specified by the orthonormal frame $\{e_1, e_2\}$. That is, the point y can be realized as $y = Q + se_1 + re_2$. Since the vector field u is a parallel laminar flow near $\partial\Omega$, the streamline of u which passes through y should be a circular arc with radius $|\overline{Cy}|$ and centered at C also. Here, $|\overline{Cy}|$ is length of the line segment \overline{Cy} . That is, the distance between C and y . Moreover, we have

$$|u(y)| = h(|\overline{Cy}| - \delta)$$

with δ to be the radius of the circular arc $\varphi([\bar{s}_1, \bar{s}_2])$, and $h(r) = \alpha_1 r - \frac{\alpha_2}{2} r^2$. So, in order to compute $|u(y)|$, it is enough to compute $|\overline{Cy}|$ by means of geometry. Let L_x to be the straight line which passes through Q and is parallel to the e_1 direction. That is L_x is the new x -axis of the new coordinate system. First, let A to be the unique point on L_x such that the line segment \overline{QA} is perpendicular to \overline{Ay} . Let D to be the point of intersection between the line segment \overline{Cy} and the line L_x . Observe that $|\overline{Ay}| = r$, $|\overline{QA}| = s$, and $|\overline{CQ}| = \delta$. We first compute $|\overline{QD}|$ and $|\overline{DA}|$ through the following observation. Since $\triangle DAy$ and $\triangle DQC$ are similar triangles, it follows that

$$\frac{|\overline{DA}|}{r} = \frac{|\overline{QD}|}{\delta}.$$

Hence, by substituting $|\overline{DA}| = (r/\delta)|\overline{QD}|$ into the equation $s = |\overline{QD}| + |\overline{DA}|$, it follows that

$$|\overline{DA}| = \frac{rs}{\delta + r} \quad \text{and} \quad |\overline{QD}| = \frac{\delta s}{\delta + r}.$$

This further gives

$$|\overline{Dy}| = r \left\{ 1 + \left(\frac{s}{\delta + r} \right)^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad |\overline{DC}| = \delta \left\{ 1 + \left(\frac{s}{\delta + r} \right)^2 \right\}^{\frac{1}{2}}.$$

Hence, we have

$$|\overline{Cy}| = |\overline{Dy}| + |\overline{DC}| = \{(\delta + r)^2 + s^2\}^{\frac{1}{2}}.$$

As a result, we have $|u(p)| = h(\{(\delta + r)^2 + s^2\}^{\frac{1}{2}} - \delta)$, and that

$$\begin{aligned} v^1 &= h(\{(\delta + r)^2 + s^2\}^{\frac{1}{2}} - \delta) \cdot \cos(\theta(y)) \\ v^2 &= h(\{(\delta + r)^2 + s^2\}^{\frac{1}{2}} - \delta) \cdot \sin(\theta(y)), \end{aligned}$$

where, $\theta(y)$ is defined by the inscribed angle between \overline{Dy} and \overline{Ay} . Since $\triangle DAy$ is a right angled triangle, it follows that

$$\begin{aligned} \sin(\theta(y)) &= -\sin(|\theta(y)|) = -\frac{|\overline{DA}|}{|\overline{Dy}|} = -\frac{s}{\{(\delta + r)^2 + s^2\}^{\frac{1}{2}}} \\ \cos(\theta(y)) &= \cos(|\theta(y)|) = \frac{\delta + r}{\{(\delta + r)^2 + s^2\}^{\frac{1}{2}}}. \end{aligned} \tag{2.10}$$

Hence, it follows that

$$\begin{aligned} v^1 &= v^1(Q + se_1 + re_2) = h(\{(\delta + r)^2 + s^2\}^{\frac{1}{2}} - \delta) \cdot \frac{\delta + r}{\{(\delta + r)^2 + s^2\}^{\frac{1}{2}}} \\ v^2 &= v^2(Q + se_1 + re_2) = -h(\{(\delta + r)^2 + s^2\}^{\frac{1}{2}} - \delta) \cdot \frac{s}{\{(\delta + r)^2 + s^2\}^{\frac{1}{2}}}. \end{aligned} \tag{2.11}$$

Therefore we obtain, through direct computation that,

$$\partial_r^2 v^1 = -\alpha_2, \quad \partial_r^2 v^2 = 0, \quad \partial_s^2 v^1|_{s=0} = \frac{1}{r + \delta} \left(\alpha_1 - \alpha_2 r - \frac{h(r)}{r + \delta} \right) \quad \text{and} \quad \partial_s^2 v^2 = 0.$$

From (1.5), we see $u(\gamma_{\Phi(0,r)}(s)) = u(\Phi(s, r)) = h(r)$. Thus the material differentiation can be calculated as

$$\begin{aligned} ((u \cdot \nabla)u)(\gamma_{\Phi(0,r)}(\bar{s})) &= \partial_s (u(\gamma_{\Phi(0,r)}(s)))|_{s=\bar{s}} \\ &= |u(\gamma_{\Phi(0,r)}(s))| \partial_s \left(\frac{u}{|u|}(\gamma_{\Phi(0,r)}(s)) \right) |_{s=\bar{s}} \\ &\quad + \partial_s |u(\gamma_{\Phi(0,r)}(s))| \left(\frac{u}{|u|}(\gamma_{\Phi(0,r)}(s)) \right) |_{s=\bar{s}} \\ &= |u(\gamma_{\Phi(0,r)}(s))| \partial_s \left(\frac{u}{|u|}(\gamma_{\Phi(0,r)}(s)) \right) |_{s=\bar{s}} \\ &= -\frac{1}{r + \delta} h(r) (\partial_s \varphi)^\perp(\bar{s}). \end{aligned}$$

Note that the above decomposition (into “curvature part” and “acceleration part”) is already done in [1]. Therefore

$$(2.12) \quad \nu \Delta u(\Phi(\bar{s}, r)) - ((u \cdot \nabla)u)(\Phi(\bar{s}, r)) = P(r) \partial_s \varphi(\bar{s}) + P^\perp(r) (\partial_s \varphi)^\perp(\bar{s})$$

for $\bar{s} \in [\bar{s}_1, \bar{s}_2]$, where

$$(2.13) \quad P(r) = \nu \left(\frac{\alpha_1}{r + \delta} - \frac{\alpha_2 r}{r + \delta} - \frac{h(r)}{(r + \delta)^2} - \alpha_2 \right), \quad P^\perp(r) := \frac{h(r)}{r + \delta}.$$

In order to see the pressure precisely, we need the following “normalized pressure-line” and “boundary of the level set of the pressure”. Let

$$\partial_s q_a(s) = \frac{\nabla p}{|\nabla p|}(q_a(s)), \quad \partial_r q_a^\perp(r) = \left(\frac{\nabla p}{|\nabla p|} \right)^\perp(q_a^\perp(r)) \quad \text{for } |\nabla p| \neq 0$$

with $q_a(0) = a$ and $q_a^\perp(0) = a$. Moreover let us define a re-parametrized q as follows:

Definition 2.1. (Poincaré map on the pressure lines.) For sufficiently small $\epsilon > 0$, let s_{min} be the minimum of $s' > 0$ for which there exists $\tau = \tau(s')$ such that

$$q_{\Phi(s,r)}(s') = q_{\Phi(s+\epsilon,r)}^\perp(\tau(s')).$$

Let $\eta(\epsilon) = \eta(\epsilon, s, r) = q_{\Phi(s,r)}(s_{min})$. We denote $\cup^\epsilon \eta(\epsilon')$ as $\cup_{\epsilon'=0}^\epsilon \eta(\epsilon', s, r)$.

Now we consider the level set of p precisely. We choose the following finite points: $\{\eta(\epsilon'(0)), \eta(\epsilon'(1)), \eta(\epsilon'(2)), \dots, \eta(\epsilon'(N-1)), \eta(\epsilon'(N))\}$ in order to satisfy that $\epsilon'(0) = 0$, $\epsilon'(N) = \epsilon$ and $|\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell-1))|$ ($\ell = 1, \dots, N$) are the same value. Then we set ∇p (in the discrete setting) from the level set of p

$$\nabla p(\eta(\epsilon'(\ell))) \approx \frac{p(\eta(\epsilon'(\ell))) - p(\eta(\epsilon'(\ell+1)))}{|\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))|} \frac{\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))}{|\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))|}$$

and let \vec{t} be a tangent vector defined as

$$\vec{t}(\epsilon'(\ell)) = \frac{\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))}{|\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))|}.$$

Then we have in the discrete setting (piecewise linear approximation):

$$\begin{aligned} \int_{\cup^\epsilon \eta(\epsilon')} \nabla p \cdot \vec{t} &= \lim_{N \rightarrow \infty} \sum_{\ell=1}^N \nabla p(\epsilon'(\ell)) \cdot \vec{t}(\epsilon'(\ell)) |\eta(\epsilon'(\ell)) - \eta(\epsilon'(\ell+1))| \\ &= p(\eta(\epsilon, s, r)) - p(\Phi(s, r)). \end{aligned}$$

Therefore

$$(2.14) \quad \begin{aligned} |\nabla p(\Phi(s, r))| &= \lim_{\epsilon \rightarrow 0} \frac{1}{|\cup^\epsilon \eta(\epsilon')|} \left| \int_{\cup^\epsilon \eta(\epsilon')} \nabla p \cdot \vec{t} \right| \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{|\cup^\epsilon \eta(\epsilon')|} |p(\eta(\epsilon, s, r)) - p(\Phi(s, r))|. \end{aligned}$$

Since the direction of ∇p , namely $\frac{\nabla p}{|\nabla p|}$ is determined by $\frac{\Delta u}{|\Delta u|}$ and $\frac{\partial_s u(\gamma(s))}{|\partial_s u(\gamma(s))|}$, and u has a parallel laminar flow, we see that

$$\left\langle \frac{\nabla p}{|\nabla p|}(\Phi(s, r)), \frac{\partial_s \Phi(s, r)}{|\partial_s \Phi(s, r)|} \right\rangle$$

is independent of s . Thus there are positive $\hat{s}(r)$ and $\hat{r}(r)$ (independent of s) such that

$$p(\Phi(s, r)) = p(q_{\Phi(s, r)}^\perp(0)) = p(q_{\Phi(s, r)}^\perp(\hat{r}(r))) = p(\Phi(s - \hat{s}(r), 0))$$

and

$$p(\eta(\epsilon, s, r)) = p(q_{\Phi(s+\epsilon, r)}^\perp(0)) = p(q_{\Phi(s+\epsilon, r)}^\perp(\hat{r}(r))) = p(\Phi(s + \epsilon - \hat{s}(r), 0)).$$

Since

$$\nabla p(\Phi(s, 0)) = \nu \Delta u(\Phi(s, 0)) = \nu \left(\frac{\alpha_1}{\delta} - \alpha_2 \right) \partial_s \varphi$$

derived from the boundary condition, we see

$$\begin{aligned} |p(\Phi(s + \epsilon - \hat{s}(r), 0)) - p(\Phi(s - \hat{s}(r), 0))| &= \left| \int_0^\epsilon \nabla p(\Phi(s', 0)) \cdot \varphi(s') \, ds' \right| \\ &= \epsilon \nu \left| \frac{\alpha_1}{\delta} - \alpha_2 \right|. \end{aligned}$$

Thus we have the following pressure estimate:

$$|\nabla p(\Phi(s, r))| = \lim_{\epsilon \rightarrow 0} \frac{\epsilon \nu |\alpha_1/\delta - \alpha_2|}{|\cup^\epsilon \eta(\epsilon')|}.$$

On the other hand, since $\theta((\partial_s \Phi)(s + \epsilon)) > \theta(\partial_s q_{\Phi(s, r)}(s + \epsilon))$ for any ϵ , there is a negative number $\tau = \tau(\epsilon, s, r) < 0$ such that $q_{\Phi(s+\epsilon, r)}^\perp(\tau) = \eta(\epsilon, s, r)$. Thus $D_\epsilon := \cup_{\epsilon'=0}^\epsilon \cup_{r'=\tau(\epsilon', s, r)}^0 q_{\Phi(s+\epsilon', r)}^\perp(r')$ is well defined and $\lim_{\epsilon \rightarrow 0} D_\epsilon/|D_\epsilon|$ is a right triangle since $\langle (\partial_s \Phi)(s, r), ((\partial_\epsilon \eta)(0, s, r)) \rangle$ does not change for any $\epsilon > 0$ and $\langle \partial_\tau q_{\Phi(s+\epsilon, r)}^\perp(\tau), (\partial_\epsilon \eta)(\epsilon, s, r) \rangle = 0$ for any $\epsilon > 0$. It means that angles of the three corners do not change for any ϵ and one of them is $\pi/2$. More precisely, among the three sides of the triangular region D_ϵ , the side $\cup_{0 \leq \epsilon' \leq \epsilon} \eta(\epsilon', s, r)$, which is a part of the pressure line passing through $\Phi(s, r)$, is *perpendicular to* the level set of the function p , which passes through $\Phi(s + \epsilon, r)$, *at the point of intersection* $\eta(\epsilon, s, r)$. In other words, we have the following observation.

- the right angle $\frac{\pi}{2}$ of the triangle D_ϵ is located at the vertex $\eta(\epsilon, s, r)$, which is the point of intersection between the side $\cup_{0 \leq \epsilon' \leq \epsilon} \eta(\epsilon', s, r)$ and the level set of p passing through $\Phi(s + \epsilon, r)$.

So, the arc segment $|\cup_{s \leq s' \leq s+\epsilon} \Phi(s', r)|$ should be the *longest side* of the triangle D_ϵ whose *opposite angle* in D_ϵ is the *right angle* $\frac{\pi}{2}$ located at the vertex $\eta(\epsilon, s, r)$ of D_ϵ . According to the above reasoning, it is not hard to see that

$$(2.15) \quad \frac{|P(r)|}{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}} = \lim_{\epsilon \rightarrow 0^+} \frac{|\cup_{0 \leq \epsilon' \leq \epsilon} \eta(\epsilon', s, r)|}{|\cup_{s \leq s' \leq s+\epsilon} \Phi(s', r)|}.$$

In accordance with the relation in (2.15), it follows that

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{|\cup_{0 \leq \epsilon' \leq \epsilon} \eta(\epsilon', s, r)|} &= \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{|\cup_{s \leq s' \leq s+\epsilon} \Phi(s', r)|} \cdot \frac{|\cup_{s \leq s' \leq s+\epsilon} \Phi(s', r)|}{|\cup_{0 \leq \epsilon' \leq \epsilon} \eta(\epsilon', s, r)|} \\
&= \frac{\epsilon}{(\frac{\delta+r}{\delta} |\cup_{s \leq s' \leq s+\epsilon} \Phi(s', 0)|)} \cdot \frac{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}}{|P(r)|} \\
&= \left(\frac{\epsilon}{(\frac{\delta+r}{\delta})\epsilon} \right) \frac{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}}{|P(r)|} \\
&= \left(\frac{\delta}{\delta+r} \right) \frac{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}}{|P(r)|}.
\end{aligned}$$

Hence, it follows from relation (2.14) and the above that

$$|\nabla p(\Phi(s, r))| = \nu \left| \frac{\alpha_1}{\delta} - \alpha_2 \right| \left(\frac{\delta}{\delta+r} \right) \frac{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}}{|P(r)|}.$$

However, since we naturally have $|\nabla p(\Phi(s, r))| = [(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}$, it follows from the above relation that

$$\nu \left| \frac{\alpha_1}{\delta} - \alpha_2 \right| \left(\frac{\delta}{\delta+r} \right) \frac{[(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}}{|P(r)|} = [(P(r))^2 + (P^\perp(r))^2]^{\frac{1}{2}}.$$

The above expression, together with the expression $P(r) = \nu \left(\frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2 \right)$ directly implies that

$$(2.16) \quad \left| \frac{\alpha_1}{\delta} - \alpha_2 \right| \left(\frac{\delta}{\delta+r} \right) = \left| \frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2 \right|.$$

Our task here is to derive a contradiction from the relation in (2.16). First, let us deal with the case in which $|\frac{\alpha_1}{\delta} - \alpha_2| > 0$. In this case, since $(\frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2) \rightarrow \frac{\alpha_1}{\delta} - \alpha_2$, as $r \rightarrow 0^+$, it follows that $(\frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2)$ will have the same sign as that of $\frac{\alpha_1}{\delta} - \alpha_2$, for all sufficiently small $r > 0$. That is, for all $r > 0$ to be sufficiently small, the following assertion holds.

$(\frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2)$ is positive if and only if $\frac{\alpha_1}{\delta} - \alpha_2$ is positive.

As a result, the relation in (2.16) can be rephrased as

$$(2.17) \quad \left(\frac{\alpha_1}{\delta} - \alpha_2 \right) \left(\frac{\delta}{\delta+r} \right) = \frac{\alpha_1}{r+\delta} - \frac{\alpha_2 r}{(r+\delta)} - \frac{h(r)}{(r+\delta)^2} - \alpha_2.$$

However, by some algebraic computation, relation as given in (2.17) is equivalent to saying that the following relation holds for all $r > 0$ sufficiently close to 0,

$$\alpha_2 \left(\frac{\delta}{\delta+r} - 1 \right) = \frac{h(r)}{(\delta+r)^2} + \frac{\alpha_2 r}{r+\delta}.$$

However, the above relation says that, for $r > 0$ to be sufficiently small, the left hand side $\alpha_2 [\frac{\delta}{\delta+r} - 1]$, which is strictly negative, equals the right hand side $\frac{h(r)}{\delta+r} + \frac{\alpha_2 r}{(r+\delta)}$, which is strictly positive. This is *absurd*, and hence we have arrive at a contradiction in this case.

The case of $\frac{\alpha_1}{\delta} - \alpha_2 = 0$ is even easier to handle, since in this case, relation (2.16) immediately gives

$$0 = \frac{\alpha_1}{\delta + r} - \frac{\alpha_2 r}{r + \delta} - \frac{h(r)}{(\delta + r)^2} - \alpha_2 = \frac{\delta \alpha_2}{\delta + r} - \frac{h(r)}{(\delta + r)^2} - \alpha_2$$

which eventually leads to the relation $\alpha_2[\frac{\delta}{\delta+r} - 1] = \frac{h(r)}{(\delta+r)^2} + \frac{\alpha_2 r}{(r+\delta)}$, which is again absurd. So, we also have a contradiction.

Proof of the second main theorem (Non-stationary case.) Let $p(x) := p(x, t)|_{t=0}$ and $u(x) := u(x, t)|_{t=0}$. From (2.13), we see that

$$\Delta u(\Phi(s, r)) = \nu \left(\frac{\alpha_1}{r + \delta} - \frac{\alpha_2 r}{r + \delta} - \frac{h(r)}{(r + \delta)^2} - \alpha_2 \right) \partial_s \varphi(s).$$

Thus we have

$$(2.18) \quad \nabla p(\Phi(s, r))|_{r=0} = \nu \Delta u(\Phi(s, r))|_{r=0} = \nu \left(\frac{\alpha_1}{\delta} - \alpha_2 \right) \partial_s \varphi(s)$$

derived from the Dirichlet boundary condition. In order to estimate ∇p only using smoothness and (2.18), we use “normalized pressure-line” and “boundary of the level set of the pressure” defined in the stationary case. The key is to estimate $|\cup_{\epsilon'=0}^{\epsilon} \zeta(\epsilon')|$ (we define it later) which is similar to $|\cup_{\epsilon'=0}^{\epsilon} \eta(\epsilon')|$ defined in the stationary case.

Definition 2.2. Here we give three definitions which are needed in a geometric observation.

- Let $\hat{s} = \hat{s}(s, r)$ and $\hat{r} = \hat{r}(s, r)$ be such that

$$q_{\varphi(s)}^{\perp}(\hat{r}) = \Phi(\hat{s}, r) \quad \text{for } s \in [\bar{s}_1, \bar{s}_2].$$

- For sufficiently small $\epsilon > 0$, let s_{min} be the minimum of $s' > 0$ for which there exists $\tau = \tau(s')$ such that $q_F(s') = q_{\varphi(s+\epsilon)}^{\perp}(\tau(s'))$, where $F = q_{\varphi(s)}^{\perp}(\hat{r}(s, r))$. Let $\zeta(\epsilon) = \zeta(\epsilon, s, r) = q_F(s_{min})$. We denote $\cup_{\epsilon'=0}^{\epsilon} \zeta(\epsilon')$ as $\cup_{\epsilon'=0}^{\epsilon} \zeta(\epsilon', s, r)$.
- Let $\hat{s} = \hat{s}(\epsilon, s, r)$ and $\hat{r} = \hat{r}(\epsilon, s, r)$ be such that $\zeta(\epsilon, s, r) = \Phi(\hat{s}, \hat{r})$.

Due to the smoothness of p and (2.18), we can estimate \hat{s} and \hat{r} as

$$(2.19) \quad s - cr^2 \leq \hat{s} \leq s + cr^2 \quad \text{and} \quad s + \epsilon - c_1 \hat{r}^2 \leq \hat{s} \leq s + \epsilon + c_2 \hat{r},$$

where c is a positive constant independent of ϵ, s, r and \hat{r} . Let us set $\hat{D}_{\epsilon, r}$ (which is similar to D_{ϵ} in the stationary case but different) as follows:

$$\hat{D}_{\epsilon, r} := \cup_{\epsilon'=0}^{\epsilon} \cup_{r'=0}^r \zeta(\epsilon', s, r').$$

Since all of the four angles are $\pi/2$,

$$\lim_{\substack{\epsilon, r \rightarrow 0 \\ r/\epsilon = \text{const.}}} \frac{\hat{D}_{\epsilon, r}}{|\hat{D}_{\epsilon, r}|}$$

is a rectangular. Thus we can see that for sufficiently small $\hat{\epsilon} > 0$, there are R, \hat{R} and $\hat{\hat{R}}$ such that

$$(2.20) \quad (1 - \hat{\epsilon})r \leq \hat{\hat{R}} \leq (1 + \hat{\epsilon})r$$

for $r < R$ with $r/\epsilon = \text{const.}$, $\hat{r} < \hat{R}$ and $\hat{\hat{r}} < \hat{\hat{R}}$. From (2.19), we have

$$s + \epsilon - c(1 + \hat{\epsilon})^2 r^2 \leq \hat{\hat{s}} \leq s + \epsilon + c(1 + \hat{\epsilon})^2 r^2$$

and then

$$\epsilon - cr^2 - c(1 + \hat{\epsilon})r^2 \leq \hat{\hat{s}} - \hat{s} \leq \epsilon + cr^2 + c(1 + \hat{\epsilon})r^2.$$

Due to the smoothness of p and (2.18), we also see that

$$1 - cr^2 \leq \left\langle \left(\frac{\nabla p}{|\nabla p|} \right) (\Phi(s, r)), \partial_s \varphi(s) \right\rangle \leq 1.$$

By using a piecewise linear approximation, we can estimate $|\cup^\epsilon \zeta(\epsilon')|$ as

$$|\cup^\epsilon \zeta(\epsilon')| = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{\sum_{k=0}^n \Delta r_k + r}{r \cos \theta(\Phi(\hat{s} + n\Delta s, r + \sum_{k=0}^n \Delta r_k))} \Delta s,$$

where $\Delta r_0 = 0$, $\Delta r_k = \tan \theta(\Phi(\hat{s} + (k-1)\Delta s, r + \sum_{k'=1}^{k-1} \Delta r_{k'}))$ for $k \geq 1$, $\theta(\Phi(s, r))$ is defined as the angle between $(\nabla p/|\nabla p|)(\Phi(s, r))$ and $\partial_s \varphi(s)$, and we choose Δs in order to satisfy

$$N\Delta s = |\cup_{s'=s}^{\hat{s}} \Phi(s', r)| = \frac{\delta + r}{\delta} (\hat{s} - s).$$

Note that Δs tends to 0 as $N \rightarrow \infty$. It means that Δr_k tends to 0 as $N \rightarrow \infty$ for fixed k . Then we see that

$$\lim_{n \rightarrow \infty, N \rightarrow \infty} \sum_{k=0}^n \Delta r_k + r = \hat{\hat{r}}.$$

By the above estimates, we have the following lower and upper bound of $|\cup^\epsilon \zeta(\epsilon')|$:

$$(2.21) \quad \begin{aligned} |\cup^\epsilon \zeta(\epsilon')| &\geq (1 - \hat{\epsilon}) \frac{r + \delta}{\delta} (\epsilon - cr^2 - c(1 + \hat{\epsilon})^2 r^2), \\ |\cup^\epsilon \zeta(\epsilon')| &\leq \frac{1 + \hat{\epsilon}}{1 - cr^2} \cdot \frac{r + \delta}{\delta} (\epsilon + cr^2 + c(1 + \hat{\epsilon})^2 r^2) \end{aligned}$$

for sufficiently small ϵ and r with $\epsilon/r = \text{const.}$ Note that we can take $\hat{\epsilon} > 0$ arbitrarily small. We have from the explicit representation of the gradient of the pressure (here we consider the case $\alpha_1/\delta - \alpha_2 > 0$):

$$\begin{aligned} \langle \nabla p(\varphi(s)), \partial_s \varphi(s) \rangle &= |\nabla p(\varphi(s))| \left\langle \frac{\nabla p}{|\nabla p|}(\varphi(s)), \varphi(s) \right\rangle \\ &= |\nabla p(\Phi(\hat{s}, r))| \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{|\cup^\epsilon \zeta(\epsilon')|} \left| \int_{\cup^\epsilon \zeta(\epsilon')} \nabla p(x) \cdot \vec{t} \, dx \right| \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{|\cup^\epsilon \zeta(\epsilon')|} |p(\varphi(s + \epsilon)) - p(\varphi(s))| \\ &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon \nu (\alpha_1/\delta - \alpha_2)}{|\cup^\epsilon \zeta(\epsilon')|}. \end{aligned}$$

Even in the case $\alpha_1/\delta - \alpha_2 \leq 0$, we have the same estimate as above. Recall that

$$\langle \nu \Delta u, \partial \varphi(s) \rangle = \nu \left(\frac{\alpha_1}{r + \delta} - \frac{\alpha_2 r}{(r + \delta)} - \frac{h(r)}{(r + \delta)^2} - \alpha_2 \right)$$

and

$$\langle u_0, \partial_s \varphi(s) \rangle = \alpha_1 r - \frac{\alpha_2}{2} r^2.$$

We already have the lower and upper bounds of $|\cup^\epsilon \zeta(\epsilon')|$ (see (2.21)), we have that

$$\begin{aligned} \frac{\langle D_t u, \partial_s \varphi(s) \rangle}{\langle u_0, \partial_s \varphi(s) \rangle} &= \frac{\langle (\nu \Delta u - \nabla p), \partial_s \varphi(s) \rangle}{\langle u_0, \partial_s \varphi(s) \rangle} \\ &= \lim_{\substack{\epsilon, r \rightarrow 0 \\ \epsilon/r = \text{const.}}} \frac{\langle \nu \Delta u(\Phi(s, r)), \partial_s \varphi(s) \rangle - (\epsilon \nu (\alpha_1 / \delta - \alpha_2) / |\cup^\epsilon \zeta(\epsilon')|)}{\langle u_0(\Phi(s, r)), \partial_s \varphi(s) \rangle} \\ &= -\frac{\nu \alpha_2}{\delta \alpha_1} - \frac{\nu}{\delta^2} \end{aligned}$$

for $s \in [\bar{s}_1, \bar{s}_2]$. Therefore we have the the desired estimate.

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REFERENCES

- [1] D. Chae, P. Constantin and J. Wu, *Deformation and symmetry in the inviscid SQG and the 3D Euler equations*. submitted.
- [2] D. G. Crighton, *The Kutta condition in unsteady flow*. Ann. Rev. Fluid Mech. 17 (1985), 411–445
- [3] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Tracts in Natural Philosophy, (1994).
- [4] Y. Giga, *A remark on a Liouville problem with boundary for the Stokes and the Navier-Stokes equations*, Dept. Math, Hokkaido Univ. EPrints Server, no. 989.
- [5] Y. Giga, K. Inui and S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*, Advances in fluid dynamics, Quad. Mat., 4, Dept. Math., Seconda Univ. Napoli, Caserta (1999), 27–68 .
- [6] P. S. Jackson and J. C. R. Junt, *Turbulent wind flow over a low hill*. Quart. J. R. Met. Soc. 101 (1975), 929–955.
- [7] J. Kato, *The uniqueness of nondecaying solutions for the Navier-Stokes equations*, Arch. Ration. Mech. Anal. 169 (2003), 159–175.
- [8] T. Kobayashi, *The relation between stationary and periodic solutions of the Navier-Stokes equations in two or three dimensional channels*. J. Math. Kyoto Univ. 49-2 (2009), 307–323.
- [9] P. Konieczny, *Thorough analysis of the Oseen system in 2D exterior domains*. Math. Methods Appl. Sci. 32 (2009), 1929–1963.
- [10] J. Leray, *Etude de diverses equations integrales non lineaires et de quelques problemes que pose l'Hydrodynamique*. J. Math. Pures. Appl. 12 (1933), 1–82.
- [11] T. Ma and S. Wang, *Boundary layer separation and structural bifurcation for 2-D incompressible fluid flows. Partial differential equations and applications*. Discrete Contin. Dyn. Syst. 10 (2004), 459–472.
- [12] Y. Maekawa, *Solution formula for the vorticity equations in the half plane with application to high vorticity creation at zero viscosity limit*. Dept. Math, Hokkaido Univ. EPrints Server, no. 992.
- [13] E. Nakai and T. Yoneda, *Riesz transforms on generalized Hardy spaces and a uniqueness theorem for the Navier-Stokes equations*. Hokkaido Math. J. 40 (2011), 67–88.
- [14] T. Uchida, K. Sugitani and Y. Ohya *Evaluation on wind characteristics around a steep simple terrain in a uniform flow (case of a two dimensional ridge model)*. J. of Wind Engineering, JAWE, 29 (2004), 35–43

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